On a formalization of Gromov's h-principle and Smale's sphere eversion theorem in Lean

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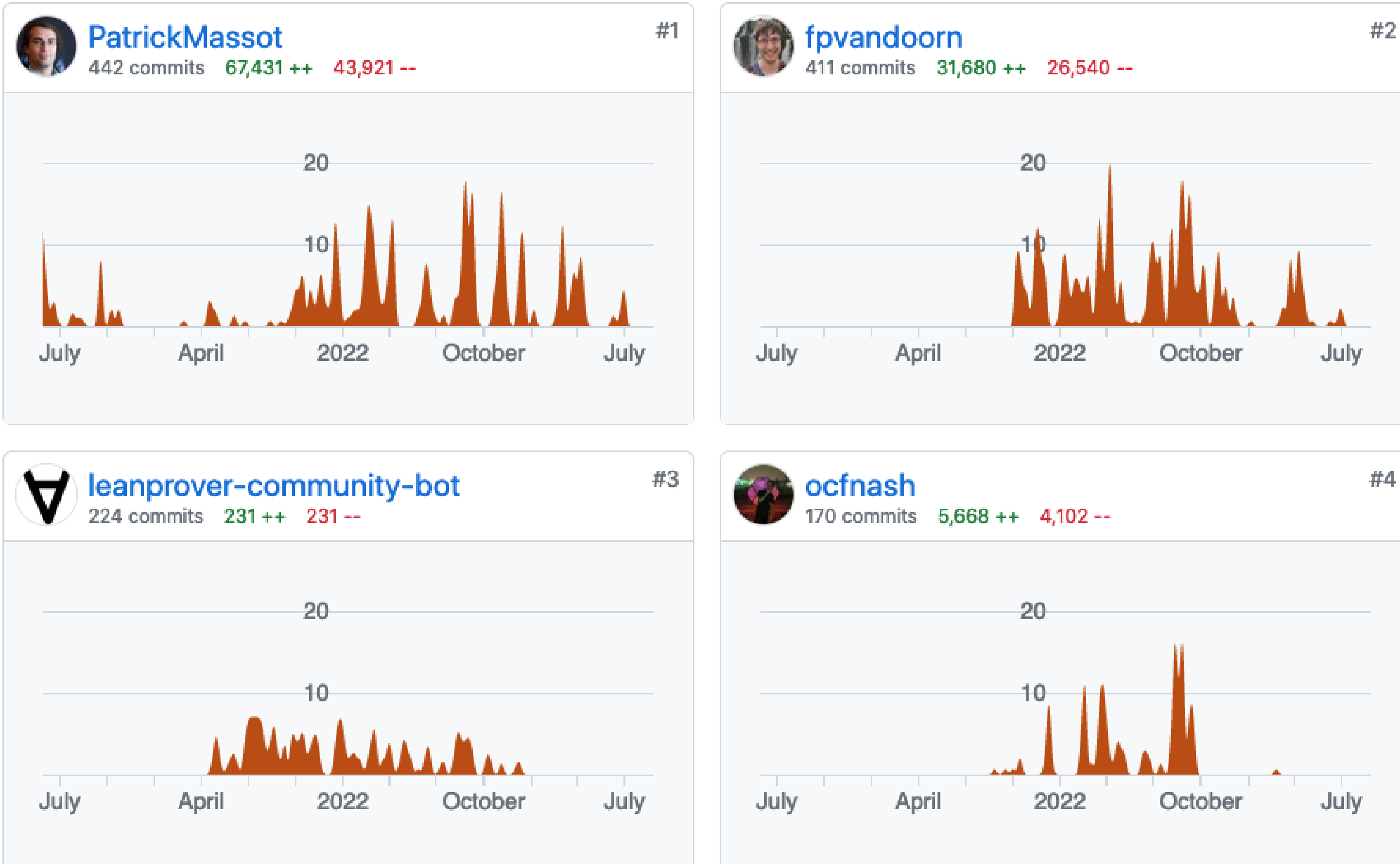


- Project completed in December 2022

- About a year with three people working part-time

Why formalise the sphere eversion theorem?

- It's fun.
- It's beautiful mathematics.
- It's a corollary of a very useful theorem of Gromov which belongs in our `mathlib` library.
- Differential topology is a relatively unexplored and underrepresented subject in formalisation.
- Test the limits (if any) of contemporary computer-formalisation technology.

Immersions

- We care about smooth manifolds so we care about maps between them.

- The derivative of a smooth map $f : M \to N$ is a (dependent) family of linear maps.

- A map f is an immersion if these linear maps are injective.

- Two basic questions:

  - Do there exist any immersions $M \to N$?

  - Are two immersions (regularly) homotopic?

From immersions to eversions

- The *inclusion map* is a natural immersion $i : S^n \to \mathbb{R}^{n+1}$.
- Assume n even for simplicity (odd case is easier).
- The *antipodal map* yields a second distinguished immersion $a : S^n \to \mathbb{R}^{n+1}$.
- Since n is even, a is orientation-reversing.
- **Question**: are i and a (regularly) homotopic?
- Such a homotopy is called an *eversion*.

Sphere eversion

- Topological obstruction gives necessary condition: $n = 2$ or $n = 6$.

- Surprising result: necessary condition is sufficient!

Theorem (Smale 1957). *There exists an eversion of* S^2*.*

```
theorem Smale : ∃ f : ℝ → 𝕊² → ℝ³,
  (cont_mdiff (𝓘(ℝ, ℝ).prod (𝓡 2)) 𝓘(ℝ, ℝ³) ∞ ↿f) ∧
  (f 0 = λ x, x) ∧
  (f 1 = λ x, -x) ∧
  ∀ t, immersion (𝓡 2) 𝓘(ℝ, ℝ³) (f t) :=
```

Enter Gromov

We formalised Smale's result as a corollary of a much more powerful theorem, due to Gromov which:

- is general: applies to maps between any manifolds $M \to N$,
- is parametric: applies to families of maps: $P \times M \to N$,
- is relative: can prescribe behaviour on a set $C \subseteq P \times M$,
- allows control: produces maps which are aribitrarily close to a given candidate map,
- generalises well beyond the concept of immersion.

Differential relations

- For simplicity consider maps of vector spaces $f : E \to F$.

- Given a relation

$$R \subseteq E \times F,$$

a function $f : E \to F$ is a solution if $(x, f(x)) \in R$ for all x.

- Given a (first-order) *differential relation*

$$R \subseteq E \times F \times \mathrm{Hom}(E, F),$$

a differentiable function $f : E \to F$ is a solution if $(x, f(x), f'(x)) \in R$ for all x.

- Example: the immersion relation is $I = \{(x, y, \phi) \mid \phi \text{ is injective}\}$.

Formal solutions and the h-principle

- A *formal solution* of a differential relation is a pair of maps $f : E \to F$ and $g : E \to \mathrm{Hom}(E, F)$ such that $(x, f(x), g(x)) \in R$ for all x.

- A formal solution is a true (holonomic) solution if $g = f'$.

- A relation is said to satisfy the **h-principle** if any formal solution is (regularly) homotopic to a true solution.

- Gromov's theorem states that if a relation is open and satisfies a **convexity condition**, it satisfies the h-principle.

- The positive-codimension immersion relation obviously satisfies the above conditions.

- Sphere eversion is thus reduced to existence of a formal solution (which is easy).

Gromov's theorem

Theorem (Gromov 1973)**.** *Let R be an open, ample differential relation for maps between two smooth manifolds M and N. Then R satisfies the parametric, relative, C^0-dense h-principle.*

I.e., given any smooth manifold P, closed set $C \subseteq P \times M$, metric on N, error function $\epsilon : M \to (0, \infty)$, and smooth family $\mathcal{F}_0$ of formal solutions to R, which is holonomic near C, then there exists a regular homotopy of smooth families of solutions $t \mapsto \widehat{\mathcal{F}}_t$ such that:

- $\widehat{\mathcal{F}}_0 = \mathcal{F}_0$,
- $\widehat{\mathcal{F}}_1$ *is holonomic,*
- $\widehat{\mathcal{F}}_t$ *is independent of t near C,*
- *the base map of $\widehat{\mathcal{F}}_t$ is ϵ-close to that of $\mathcal{F}_0$.*

Gromov's theorem in Lean

```
theorem Gromov {R : rel_mfld 𝓘(ℝ, ℝ^n) M 𝓘(ℝ, ℝ^n') M'}
  (hRample : R.ample) (hRopen : is_open R)
  {C : set (P × M)} (hC : is_closed C)
  {ε : M → ℝ} (hε : ∀ x, 0 < ε x) (hε' : continuous ε)
  (𝓕₀ : family_formal_sol 𝓘(ℝ, ℝ^d) P R)
  (hhol : ∀ᶠ (p : P × M) in 𝓝ˢ C,
    (𝓕₀ p.1).is_holonomic_at p.2) :
  ∃ 𝓕 : family_formal_sol
      (𝓘(ℝ, ℝ).prod 𝓘(ℝ, ℝ^d)) (ℝ × P) R,
    (∀ p x, 𝓕 (0, p) x = 𝓕₀ p x) ∧
    (∀ p, (𝓕 (1, p)).to_one_jet_sec.is_holonomic) ∧
    (∀ᶠ (p : P × M) in 𝓝ˢ C, ∀ t,
      𝓕 (t, p.1) p.2 = 𝓕₀ p.1 p.2) ∧
    (∀ t p x, dist ((𝓕 (t, p)).bs x) ((𝓕₀ p).bs x) ≤ ε x)
```

Why convexity

Theorem. *Let* $f : \mathbb{R} \to F$, $f_n : \mathbb{R} \to F$, *all smooth,* $\Omega \subseteq F$ *and suppose that:*

- $f_n \to f$ *in the* C^0 *topology,*
- $f_n'(s) \in \Omega$ *for all* n, s,

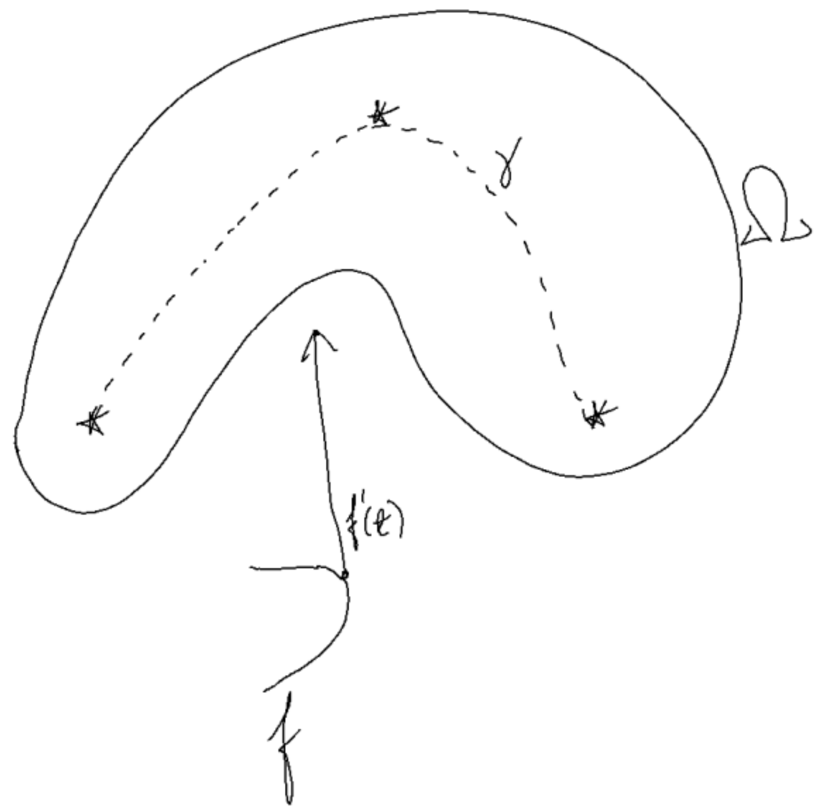
then $f'(s) \in \overline{\mathrm{Conv}(\Omega)}$ *for all* s, *where* Conv *denotes convex hull.*

Proof.

$$\frac{f(s+1/n) - f(s)}{1/n} \approx \frac{f_n(s+1/n) - f_n(s)}{1/n}$$
$$= \int_0^1 f_n'(s+u/n)du \in \overline{\mathrm{Conv}(\Omega)}$$

□

Deformation using families of loops



- We use a family loops $\gamma : E \times S^1 \to \Omega \subseteq F$ to modify f so that its derivative lies in Ω.

- We require $\overline{\gamma_x} = \int_{S^1} \gamma_x = f'(x)$ for all x in E.

Theillière corrugation (convex integration)

Theorem (Theillière, 2018). *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be* C^1 *and* $\gamma : \mathbb{R}^n \times S^1 \to \mathbb{R}^m$ *be a* C^1 *family of loops such that* $\overline{\gamma_x} = \partial_j f(x)$ *for some* $j \in 1, 2, \ldots n$. *Given* N *in* $\mathbb{R}$ *let:*

$$\widehat{f}_N(x) = f(x) + \frac{1}{N} \int_0^{Nx_j} (\gamma_x(s) - \bar{\gamma}_x)\, ds,$$

then for any compact $K \subseteq \mathbb{R}^n$ *and* $\epsilon > 0$ *if* N *is large enough we have:*

- $\|\widehat{f}_N(x) - f(x)\| < \epsilon$,
- $\|\partial_i \widehat{f}_N(x) - \partial_i f(x)\| < \epsilon$ *for* $i \neq j$,
- $\|\partial_j \widehat{f}_N(x) - \gamma_x(Nx_j)\| < \epsilon$,

for all $x \in K$.

Theillière in Lean

```
def corrugation (π : E →L[ℝ] ℝ)
  (N : ℝ) (γ : E → loop F) : E → F :=
λ x, (1/N) • ∫ t in 0..(N*π x), (γ x t - (γ x).average)

def corrugation.remainder (π : E → ℝ)
  (N : ℝ) (γ : E → loop F) : E → (E →L[ℝ] F) :=
λ x, (1/N) • ∫ t in 0..(N*π x),
  ∂₁ (λ x t, (γ x).normalize t) x t

lemma fderiv_corrugated_map (hN : N ≠ 0)
  (hγ_diff : 𝒞 1 ↿γ) {f : E → F} (hf : 𝒞 1 f) {x}
  (p : dual_pair E) (hfγ : (γ x).average = D f x p.v) :
  D (f + corrugation p.π N γ) x =
    p.update (D f x) (γ x (N*p.π x)) +
    corrugation.remainder p.π N γ x :=

lemma remainder_c0_small_on {K : set E} (hK : is_compact K)
  (hγ_diff : 𝒞 1 ↿γ) {ε : ℝ} (ε_pos : 0 < ε) :
  ∀ᶠ N in at_top, ∀ x ∈ K,
    ‖corrugation.remainder π N γ x‖ < ε :=
```

Comments on effort

The proof is a three-stage argument:

- Supply of loops: challenging but in line with expectations,

- Theillière's convex integration: challenging but in line with expectations,

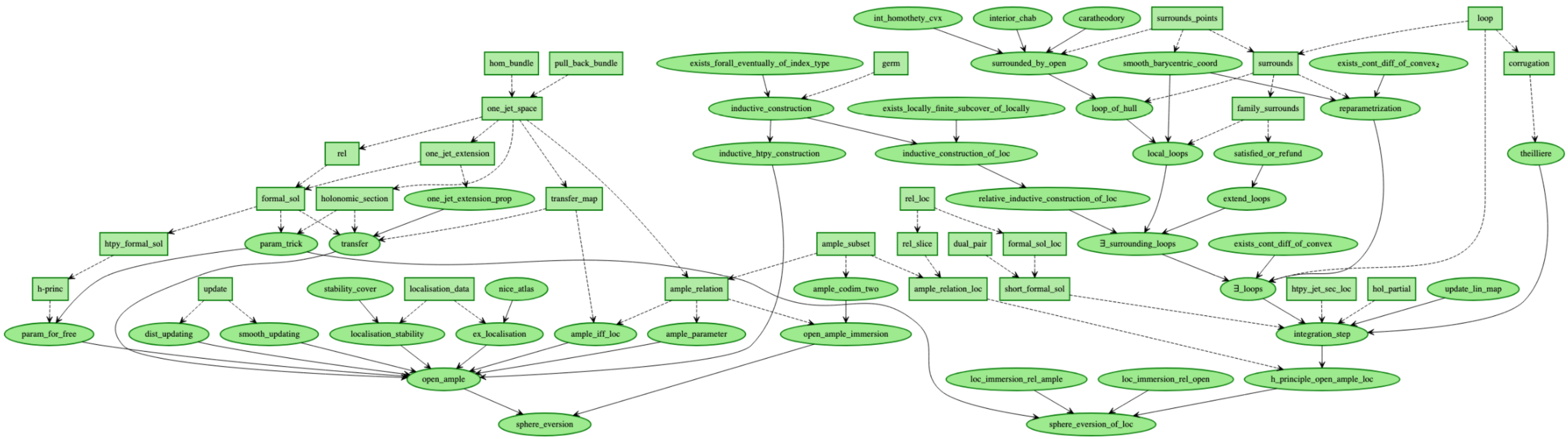
- Globalisation: challenging, beyond expectations.

Lessons learned and impact on `mathlib`

- Our manifold library works well.
- We can formalise non-trivial results in differential topology.
- Smooth fibre bundles are surprisingly tricky.
- Monolithic library essential as we draw on many subject areas.
- Significant benefit to `mathlib` (added many results on filters, point-set topology, calculus, convolutions, barycentric coordinates, convexity, bundle theory, ...).

Dependency graph

A significant aid to collaboration:



Additional resources

- We wrote a paper: van Doorn, Massot, and Nash, *Formalising the h-Principle and Sphere Eversion*, Proceedings of the 12th ACM SIGPLAN International Conference on Certified Programs and Proofs, 2023, url.
  (We might even write another.)

- Interactive website (well worth a visit):
  `https://leanprover-community.github.io/sphere-eversion`

- Previous talks:
  - van Doorn CPP 2023
  - van Doorn Lean in Lyon 2022
  - Massot Lean in Lyon 2022